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GEOMETRICAL ILLUSTRATIONS

OF THE

DIFFERENTIAL CALCULUS

BY

MORRIS BIRKBECK PELL, B.A.,

FELLOW OF SAINT JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE.

In the following pages the Author has endeavoured to illustrate geometrically the meaning of the various symbols made use of in the Differential Calculus. They are published not as putting forward any new principles or methods, but with the hope that they may render the acquisition of those already well known more easy and perfect to beginners.

The ordinary processes of the Differential Calculus are so soon acquired and so easily performed, and a fallacious idea that the principles are understood so readily entertained, that the Author imagines that anything which is calculated to turn the Student's attention more to the principles of the subject, and less to the algebraical processes, may prove of some advantage.

Let P be any point in this curve; $ON = x$, $NP = y$, its coordinates referred to the rectangular axes Ox and Oy.

Let (x) receive the small increment $NN' = dx$, and let P' be the corresponding point in the curve: draw PR' parallel to Ox, and join PP'. $P'R'$ is the increment dy of (y) corresponding to the small increment of (x). The equation (2) therefore expresses the relation between NN' or PR' and $P'R'$; and the differential coefficient $\frac{dy}{dx}$, which is their ratio, is the tangent of the angle $P'PR'$; or since dx and dy are small, PP' coincides with the tangent at P to the curve, and $\frac{dy}{dx}$ is the tangent of the angle of inclination to the axis of (x) of the line touching the curve at P.

The differential coefficient $\frac{dy}{dx}$ then is a measure of the rate of increase of (y), supposing (x) to increase uniformly. Or supposing (x) to increase in any manner, it is a measure of the rate of increase of (y) relative to that of (x).

2. Now supposing dx a constant quantity, that is independent of (x), dy is in general a function of (x) expressed by the equation

$$dy = f'(x)\, dx,$$

which also appears from the figure; for supposing NN' or dx to remain constant, the magnitude of $P'R'$ or dy will depend upon the position of the point P in the curve or upon the value of (x). If then (x) receive the increment dx, dy, in the same manner as any other function of x, will receive a corresponding increment. This is represented by d^2y, and is connected with (x) and dx by the equation

$$d^2y = \{f''(x)\, dx\}\, dx,$$

$$= f''(x)\, (dx)^2.$$

We will now consider the geometrical meaning of the quantity represented by d^2y. It is the increment which dy or $P'R'$ (fig. Art. 1) receives when x becomes $x + dx$. Take $N'N'' = dx$, and draw the corresponding ordinate $P''N''$, and $P'R''$ parallel to Ox. $P''R''$ is what $P'R'$ or dy becomes when (x) becomes $x+dx$, and d^2y therefore is the difference between the lines $P''R''$ and $P'R'$.

$\dfrac{d^2y}{dx^2}$ is a measure of the rate of increase of $\dfrac{dy}{dx}$ on the supposition of x increasing uniformly.

3. In the same manner d^3y is the increment of d^2y, considered as a function of (x), and is connected with x and dx by the equation

$$d^3y = f'''(x)\,(dx)^3.$$

If we take $N''N''' = dx$, and P''', R''' points corresponding to P' and R', then the difference $P'''R''' - P''R''$ is what d^2y becomes when (x) becomes $x+dx$, and d^3y therefore is the difference between $(P'''R''' - P''R'')$ and $(P''R'' - P'R')$ or $P'''R''' - 2P''R'' + P'R'$.

$\dfrac{d^3y}{dx^3}$ is a measure of the rate of increase of $\dfrac{d^2y}{dx^2}$, supposing x to increase uniformly.

In a similar manner may be explained the geometrical meanings of d^4y, d^5y, &c.. . . .

4. We have so far considered (x) as the independent variable or dx constant, so that d^2x, d^3x, &c. are all zero; but if we make (y) the independent variable, so that dy is constant, then dx is a function of (x). Which appears also from geometrical considerations; for if the line $P'R'$ remain constant, NN' or dx will vary for different positions of P or for different values of (x). The geometrical meanings of d^2x, d^3x, &c. may be explained in the same manner as those of d^2y, d^3y, &c.. . . .

5. The reader will have observed that in a previous Article it is stated, (x) being the independent variable, that dy is *in general* a function of (x). There is one case of exception, when (x) and (y) are connected by an equation of the form

$$y = mx + c \;\ldots\ldots\ldots\ldots\ldots\ldots\;(1),$$

so that $$dy = m\,dx:$$

in this case dy, instead of being a function of (x), is constant. Equation (1) represents a straight line, and it is manifest that for all points in the line dy is constant if dx be so. Also the tangent at any point coincides with the line itself, so that the tangent of the angle of inclination of the touching line to the axis of (x) or $\frac{dy}{dx}$ is the same for all points in the line, and therefore independent of (x).

6. We will next suppose that (x) and (y) are functions of another variable (t), so that

$$\left.\begin{aligned} x &= f(t) \\ y &= \phi(t) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Suppose (t) to be the independent variable, then

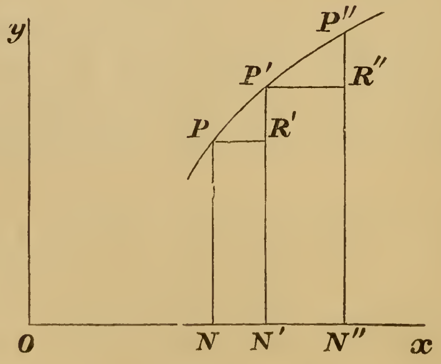
$$dx = f'(t)\,dt,$$
$$d^2x = f''(t)\,(dt)^2,$$
$$\cdots\cdots\cdots\cdots$$
$$dy = \phi'(t)\,dt,$$
$$d^2y = \phi''(t)\,(dt)^2.$$

Now the elimination of (t) between equations (1) will lead to an equation

$$y = \psi(x),$$

which will represent some curve.

Let, as before, $ON = x$, $NP = y$, be the coordinates of any



point P in the curve: take $NN' = dx$, the increment which (x)

receives when (t) becomes $t + dt$, then P' being the corresponding point in the curve, $P'R'$ is the contemporaneous increment of (y) or dy. In order to interpret geometrically the quantities d^2x and d^2y, which are the increments of dx and dy when (t) becomes $t + dt$, take $N'N'' = dx + d^2x$, which dx becomes on that supposition. Let P'' be the corresponding point in the curve, and draw $P'R''$ parallel to Ox; then, $P''R''$ being what $P'R'$ or dy becomes when (t) becomes $t + dt$, is equal to $dy + d^2y$. d^2y then is the difference between the lines $P''R''$ and $P'R'$.

d^3y may be similarly interpreted by taking $N''N''' = dx + 2d^2x + d^3x$, which is the value of $N'N''$ or $dx + d^2x$ when (t) becomes $t + dt$.

It is manifest that d^2y, d^3y, &c., found in this case by supposing (t) and not (x) to vary uniformly, are quite different from the quantities represented by the same symbols, but found on the supposition of (x) varying uniformly.

7. Now suppose that having given the equations (1) (Art. 6), it were required to find the second differential coefficient of (y) with respect to (x). We have

$$dy = \phi'(t)\,dt,$$

$$d^2y = \phi''(t)(dt)^2,$$

$$dx = f'(t)\,at;$$

we cannot however hence deduce that

$$\frac{d^2y}{dx^2} = \frac{\phi''(t)(dt)^2}{\{f'(t)\}^2(dt)^2},$$

$$= \frac{\phi''(t)}{\{f'(t)\}^2}:$$

for although this equation is of course true, remembering what is meant by d^2y, yet d^2y here represents a quantity quite different from what is conventionally understood to be represented by that symbol in the expression $\frac{d^2y}{dx^2}$. It represents the increment of dy when (t) varies uniformly; but in order to find $\frac{d^2y}{dx^2}$ in its usual sense, we must remember that d^2y is to be obtained on the supposition of (x) increasing uniformly, which in general requires that (t) do not so increase. Instead then of supposing, as in the above case, that dt is constant, so that $d^2t = 0$, we must suppose dx to be constant, and consequently $d^2x = 0$. The operation then will be as follows:

$$dx = f'(t)\,dt;$$

and since $d^2x = 0$,

$$0 = f''(t)(dt)^2 + f'(t)\,d^2t,$$

$$dy = \phi'(t)\,dt,$$

$$d^2y = \phi''(t)(dt)^2 + \phi'(t)\,d^2t.$$

Eliminating d^2t, we have

$$d^2y = \phi''(t)(dt)^2 - \frac{f''(t)}{f'(t)}\,\phi'(t)(dt)^2;$$

therefore
$$\frac{d^2y}{dx^2} = \frac{\phi''(t)f'(t) - f''(t)\,\phi'(t)}{\{f'(t)\}^3},$$

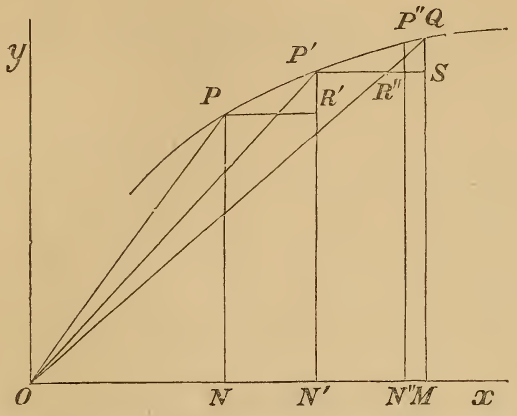
$$= \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3};$$

d^2x and d^2y on the right-hand side of the equation representing what would be the increments of dx and dy if dt were considered constant.

8. As a particular example of the above, we will suppose (xy), the coordinates of any point P in a curve, to be given as functions of (θ), the angle which the line OP makes with the axis of y.

Take $NN' = dx$, then $P'R' = dy$, and $\angle POP' = d\theta$, the corresponding increment of (θ). If now $N'N'' = dx$, and P'' be the corresponding point in the curve, $P''R'' - P'R'$ is the value of d^2y, (x) being the independent variable, and $\angle P'OP'' - \angle POP'$ is the corresponding value of $d^2\theta$. But we wish to obtain the geometrical values of d^2x and d^2y on the supposition of (θ) being the independent variable; we must therefore suppose (θ) to vary uniformly.

Draw the line OQ, making the $\angle QOP' = \angle POP'$, and cutting the curve in the point Q. Draw the ordinate QM,



and $P'S$ parallel to OX. QS is the new value of dy on the supposition of (θ) varying uniformly, and $N'M$ the new value of dx on the same supposition. We have then

$$QS - P'R' = d^2y \quad \text{and} \quad N''M = d^2x.$$

In a similar manner, if we take (r) the distance of the point P from the origin as the independent variable, and suppose (x) and (y) as given functions of (r), we may interpret the corresponding meaning of the quantities d^2x and d^2y.

9. We may otherwise illustrate the meaning of the differentials of (y), and without reference to the geometry of coordinates, in the following manner.

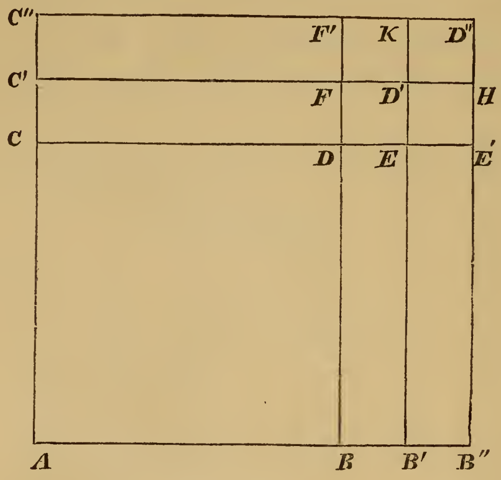
Suppose (x) to be the side AB of a square and (y) its area, then (x) and (y) are connected by the equation

$$y = x^2;$$

and supposing (x) to be the independent variable, we have

$$dy = 2x\,dx,$$
$$d^2y = 2\,(dx)^2;$$

In order to illustrate the meaning of these quantities, suppose AB to receive the increment $BB' = dx$, and let AD' be the square described on AB'. Since in the limit the area of the square DD' will vanish in comparison with the rectangles



BE and CF, dy, or the limiting value of the increment of (y) corresponding to the increment dx of (x), will be the sum of the areas of these two rectangles which is $2x\,dx$, as also appears from the analysis.

We will now proceed to consider the geometrical interpretation of d^2y. dy or the sum of the areas of the rectangles BE, CF being a function of (x) or AB, the question is, what will it become when (x) receives the increment dx? Take $B'B'' = BB'$ and complete the figure. The sum of the rectangles $B'H$ and $C'K$ is the new value of dy or $dy + d^2y$, and therefore d^2y is the sum of the squares EH and FK, which is equal to $2(dx)^2$, which agrees with the result analytically obtained.

10. We will now differentiate the equation

$$y = x^2:$$

considering y as the independent variable which gives us

$$dy = 2x\,dx,$$

$$0 = 2(dx)^2 + 2x\,d^2x;$$

or

$$dx = \frac{dy}{2x} \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$d^2x = -\frac{(dx)^2}{x} \ldots\ldots\ldots\ldots\ldots(2).$$

To interpret geometrically the meaning of these quantities, suppose the square AD to become AD', or neglecting the small square DD' to receive the increment dy equal the sum of the rectangles BE and CF; then (x) receives the corresponding increment BB', or

$$dx = \frac{\text{area } BE}{BD} = \frac{\frac{1}{2}dy}{x} = \frac{dy}{2x},$$

which agrees with equation (1).

Now dx being a function of (y) will receive an increment d^2x, when (y) is increased to $y + dy$. Let the square AD' become AD'', AD'' being such that the sum of the rectangles $B'H$ and $C'K$ is equal to that of BE and CF; $B'B''$ is the new value of dx, and our object is to find by how much it differs from BB' or dx. The rectangle $B'H = BE$, and in the limit the rectangle $EH = (dx)^2$, the difference between EE' and dx being small compared with dx. It follows that the difference between the rectangles BE and $B'E'$ is $(dx)^2$, therefore

$$(B'B'' - BB')\,x = -(dx)^2,$$

or

$$B'B'' - BB' = -\frac{(dx)^2}{x};$$

therefore the increment d^2x of BB' or dx is $-\frac{(dx)^2}{x}$, which agrees with equation (2).

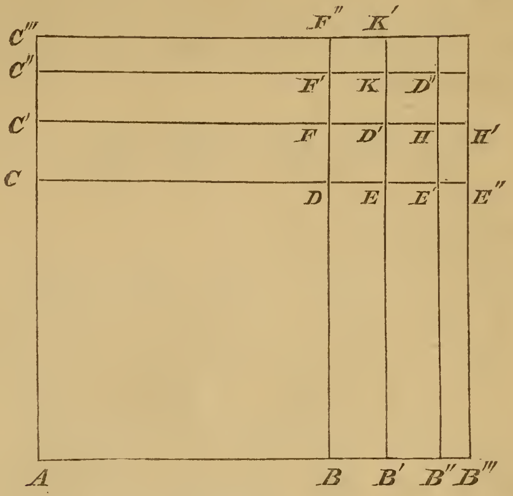
11. In the preceding Articles we have illustrated the meaning of the differentiation of the equation

$$y = x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

first considering (x), and secondly (y) as the independent variable. We will now suppose (x) to be a function of some other variable (t) considered as the independent variable. Differentiating (1) on this supposition, we have

$$dy = 2x dx,$$
$$d^2y = 2\,(dx)^2 + 2x\,d^2x,$$

d^2x and d^2y being what dx and dy respectively become when (t) becomes $t + dt$.



Take $B'B''' = dx + d^2x$, so that $B''B''' = d^2x$, then dy is the sum of the rectangles $B'D$ and $C'D$; and the value of dy when t becomes $t + dt$, or $dy + d^2y$, is the sum of the rectangles $B'''D'$ and $C'''D'$, or neglecting the rectangles $E'H'$ and $F'K'$ as being small quantities of a higher order than those retained, is the sum of the rectangles $B'''E$, $C'''F$, and of the squares EH, FK. d^2y then, being the difference between the new and the old values of dy, is the sum of the rectangles $B''E''$, $C''F''$, and of the squares EH, FK, and is therefore equal to

$$2x\,d^2x + 2\,(dx)^2;$$

which agrees with our analysis.

12. For the more complete understanding of the subject we will go through the analytical process by which dy and d^2y are found, and shew the identity of the successive steps with those in the geometrical method just employed.

If $$y = x^2,$$

then $$\delta y = (x + \delta x)^2 - x^2,$$

$$= 2x\,\delta x + (\delta x)^2,$$

and in the limit, neglecting the small quantity $(dx)^2$,

$$dy = 2xdx.$$

This is equivalent to what we did in finding dy geometrically, viz. to neglecting the small square DD' in comparison with the rectangles BE and CF.

Again,

$$d^2y = 2(x + dx)(dx + d^2x) - 2x\,dx,$$

$$= 2(dx)^2 + 2x\,d^2x + 2dx\,d^2x;$$

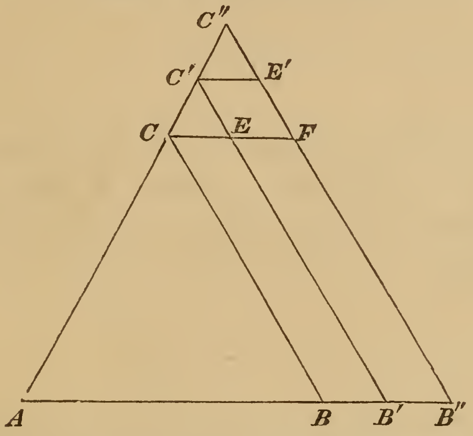
or neglecting the small quantity of the third order $2dx\,d^2x$, we have

$$d^2y = 2(dx)^2 + 2x\,d^2x.$$

The small quantity $2dx\,d^2x$ which we here neglect, is the same as the sum of the rectangles $E'H'$ and $F'K'$, which we neglected in the geometrical method of finding d^2y.

13. This part of the subject may be still farther illustrated by the following examples.

Let (x) be the length of the side AB of the equilateral triangle ABC, (y) its area, then (x) and (y) are connected by the equation



$$y = \frac{\sqrt{3}}{4}x^2;$$

and supposing (x) the independent variable,

$$dy = \frac{\sqrt{3}}{2}x\,dx,$$

$$d^2y = \frac{\sqrt{3}}{2}(dx)^2.$$

Take $BB' = dx$, and draw $B'C'$ parallel to BC, and CE to AB; then the triangle $AB'C'$ has for the length of each of its sides $x + dx$, and its area therefore is the corresponding new value of (y). We have therefore dy equal to $BB'CC$, or neglecting the small triangle CEC', dy equals the parallelogram BE; therefore

$$dy = x\,dx \sin CBB',$$

$$= \frac{\sqrt{3}}{2}\, x\,dx.$$

Again, to interpret the meaning of d^2y, take $B'B'' = dx$, draw $B''C''$ parallel to $B'C'$, and $C'E'$ to AB. The parallelogram $B'E'$ is the new position of the parallelogram BE, and its area therefore is the new value of dy, or is equal to $dy + d^2y$, so that d^2y is equal to the parallelogram EE' whose sides are each dx; therefore

$$d^2y = (dx)^2 \sin FEC',$$

$$= \frac{\sqrt{3}}{2}\,(dx)^2.$$

Considering neither (x) nor (y) as the independent variable, we might shew by the same method as that made use of in Art. 11, the different meaning to be attached on this supposition to the symbol d^2y.

Functions of two Independent Variables.

14. We shall now proceed to the consideration of functions of two independent variables.

Suppose z to be such a function of x and y, so that

$$z = f(x, y) \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Differentiating, we have

$$\frac{dz}{dx} = f'(x), \qquad \frac{dz}{dy} = f'(y).$$

These partial differential coefficients are obtained on the particular hypothesis, first of x varying and y remaining constant, and secondly of y varying and x remaining constant. The symbol dz in the two cases represents different things. In the former it denotes the increment of z when x becomes $x + dx$, y remaining constant; in the latter, the increment of z when y becomes $y + dy$, and x remains constant. If dz be the total differential of z, or the whole increment of z when x and y both vary, it may be shewn that

$$dz = \frac{dz}{dx}\,dx + \frac{dz}{dy}\,dy.$$

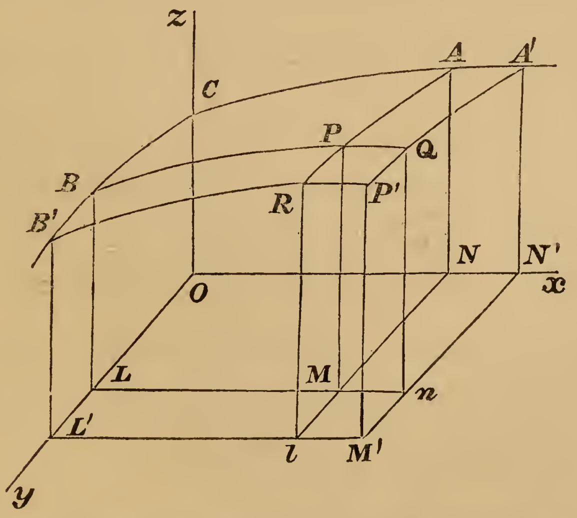
In this equation dz represents three different things. To prevent confusion, the partial differentials are sometimes expressed by the symbols $d_x z$ and $d_y z$, so that

$$dz = \frac{d_x z}{dx}\,dx + \frac{d_y z}{dy}\,dy.$$

In order to illustrate geometrically the meaning of the above analytical expressions, we shall in the first place have recourse to Geometry of Three Dimensions.

If x, y, z be the coordinates of any point in space referred to rectangular coordinate axes, then equation (1) represents some curve surface.

Let P be any point in this surface, $ON = x$, $NM = y$, $MP = z$, its coordinates referred to the system of rectangular



axes Ox, Oy, Oz; CA, CB the traces of the surface on the planes of xz and yz; AP, BP sections of the surface by planes AM, BM passing through P, and parallel to the planes yz and xz respectively. Take $NN' = dx$, and $LL' = dy$, dx and dy being supposed indefinitely small, and through N' and L' draw planes $A'M'$, $B'M'$ parallel to those of yz and xz respectively, and cutting the surface in the curves $A'P'$, $B'P'$, and the plane of xy in the lines $N'M'$ and $L'M'$.

Let us now consider the geometrical meaning of our first supposition of y remaining constant, and x varying. By this supposition, z is reduced to a function of one independent variable, and equation (1) will then express the relation between the coordinates of those points of the surface for which y has a constant value. If OL be that constant value, then those points all lie in the curve BPQ, and equation (1) will be the equation to that curve. In supposing then x to become $x + dx$, and y to remain constant, we pass from the point P to Q, and Qn is the new value of z, so that

$$Qn = z + d_x z,$$

and $d_x z$, therefore, is the difference between Qn and PM. Similarly, when we suppose y only to vary, we pass from the point P to R, and $d_y z$ represents the difference between Rl and PM.

Differentiating again, we have

$$\frac{d_x^2 z}{dx^2} = f''(x), \quad \frac{d^2_{yx} z}{dy\,dx} = f''(x, y), \quad \frac{d_y^2 z}{dy^2} = f''(y).$$

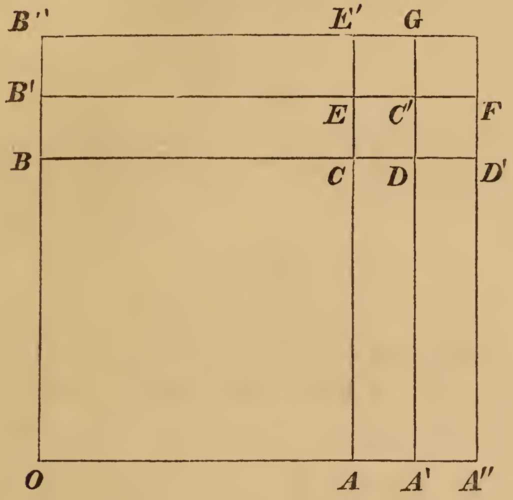
The geometrical meaning of $d_x^2 z$ and of $d_y^2 z$ may clearly be explained in the same manner as for functions of one independent variable. $d^2_{yx} z$, or the value of $d^2 z$ in the expression $\frac{d^2 z}{dy\,dx}$, is the increment which $d_x z$ receives when y becomes $y + dy$ and x remains constant. Now $d_x z = Qn - PM$; and when y becomes $y + dy$ P moves to R and Q to P', and therefore Qn becomes $P'M'$, and PM becomes Rl, and therefore $d_x z$ becomes $P'M' - Rl$; so that

$$\begin{aligned} d^2_{yx} z &= (P'M' - Rl) - (Qn - PM), \\ &= P'M' + PM - (Rl + Qn). \end{aligned}$$

The value of d^2z in the expression $\frac{d^2z}{dx\,dy}$, or as it may be written, $d^2_{xy}z$, may easily be shewn to be the same thing as $d^2_{yx}z$.

15. The following illustration may perhaps render clearer the conception of the partial differentials of a function of two independent variables.

Let the sides OA, OB of the rectangle AB be represented



by x and y respectively, and its area by z. Then we have the equation

$$z = xy \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where x and y are independent.

Differentiating this equation, we have

$$\left.\begin{array}{l} \dfrac{d_xz}{dx} = y \\[2ex] \dfrac{d_yz}{dy} = x \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2);$$

$$\left.\begin{array}{l} \dfrac{d_x^{\,2}z}{dx^2} = 0 \\[2ex] \dfrac{d^2_{yx}z}{dx\,dy} = 1 \\[2ex] \dfrac{d_y^{\,2}z}{dy^2} = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

We will now proceed to consider the geometrical meaning of these quantities. In the first place, let OA or x receive the small increment AA' or dx, OB or y remaining constant; complete the rectangle $A'B$; then the rectangle AD is the increment which the rectangle AB or z receives on the above supposition, or the quantity analytically represented by $d_x z$. Now the area of the rectangle AD is ydx, therefore

$$d_x z = ydx,$$

or

$$\frac{d_x z}{dx} = y.$$

In exactly the same manner, we might shew geometrically that

$$\frac{d_y z}{dy} = x.$$

In order to interpret the meaning of equations (3), produce OA' and OB' to A'' and B'', and make $A'A''$ and $B'B''$ equal to AA' and BB' respectively, and complete the figure.

The rectangle $A'D'$ is what AD becomes when x receives the increment dx and y remains constant, and $A'D'$ being manifestly equal to AD, the increment of AD or $d_x z$, which is analytically represented by $d_x^2 z$ is zero, and therefore

$$\frac{d_x^2 z}{dx^2} = 0,$$

and similarly,

$$\frac{d_y^2 z}{dy^2} = 0.$$

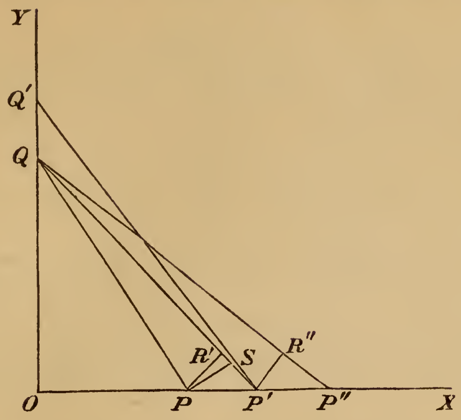
Now if x remain constant, and y vary, then the rectangle AD or $d_x z$ becomes AC', or $d_x z$ receives the increment CC' which is equal to $dy\,dx$; and this increment of $d_x z$ is represented by $d^2_{yx} z$, therefore

$$d^2_{yx} z = dy\,dx,$$

or

$$\frac{d^2 z}{dy\,dx} = 1.$$

16. Let OP and OQ be two lines perpendicular to one another, $OP = x$, $OQ = y$, and $PQ = z$; then (z) is a function



of the two independent variables (x) and (y), being connected with them by the equation

$$z = \sqrt{(x^2 + y^2)}.$$

Differentiating
$$\frac{d_x z}{dx} = \frac{x}{\sqrt{(x^2 + y^2)}},$$
$$\frac{d_y z}{dy} = \frac{y}{\sqrt{(x^2 + y^2)}}.$$

Take $PP' = dx$ the small increment of (x), and join QP': QP' is the new value of (z), and if PR' be drawn perpendicular to QP' then $P'R' = d_x z$ since $Q'R'$ is ultimately equal to QP. In a similar manner might $d_y z$ be geometrically represented.

Again, take $P'P'' = PP'$, join QP'', and draw $P'R''$ perpendicular to QP''; then $P''R''$ is what $P'R'$ or $d_x z$ becomes when (x) becomes $x + dx$, and therefore

$$d_x^2 z = P''R'' - P'R'.$$

To interpret the meaning of $d^2_{yx} z$ or the value of $d^2 z$ in the expression $\frac{d^2 z}{dy\, dx}$, take $QQ' = dy$, join $P'Q'$, and draw PS perpendicular to $P'Q'$; then $P'S$ is what $P'R'$ or $d_x z$ becomes when (y) becomes $y + dy$, and therefore

$$d^2_{yx} z = P'S - P'R':$$

$d_y^2 z$ and $d^2_{xy} z$ may be interpreted in a precisely similar manner.

17. If u be a function of two independent variables x and y expressed by the equation

$$u = f(x, y) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and x and y be given functions of two other independent variables r and θ, so that

$$x = \phi(r, \theta), \quad y = \psi(r, \theta) \ldots\ldots\ldots\ldots (2),$$

$\frac{du}{dx}$ and $\frac{du}{dy}$ may be obtained in terms of the variables r and θ, and the differential coefficients of u with respect to those variables, as follows. Differentiating (1) with respect to r and θ,

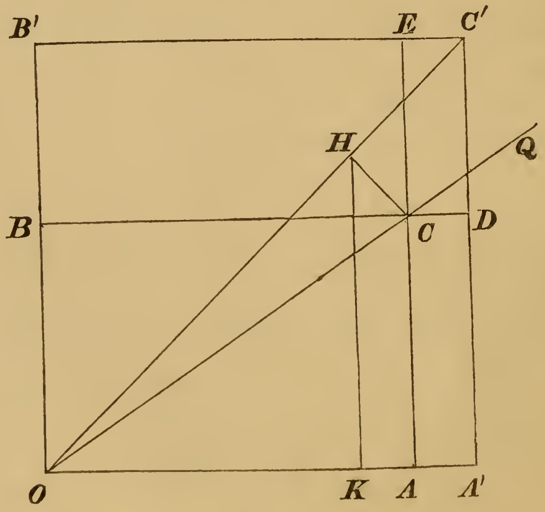
$$\frac{du}{dr} = \frac{du}{dx}\frac{dx}{dr} + \frac{du}{dy}\frac{dy}{dr},$$

$$\frac{du}{d\theta} = \frac{du}{dx}\frac{dx}{d\theta} + \frac{du}{dy}\frac{dy}{d\theta},$$

$\frac{dx}{dr}$, $\frac{dy}{dr}$, &c.... being found from equations (2); $\frac{du}{dx}$ and $\frac{du}{dy}$ may be obtained in terms of r, θ, $\frac{du}{dr}$ and $\frac{du}{d\theta}$.

We will now illustrate by means of a particular example the meaning of the various symbols involved in this process.

Let the sides OA, OB of the rectangle AB be represented



by x and y respectively, and its area by u. Let OC be represented by r, and the angle AOC by θ. Then we have the equations

$$u = xy \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x = r\cos\theta, \quad y = r\sin\theta. \ldots\ldots\ldots\ldots (2).$$

Differentiating,

$$\frac{du}{dr} = y\frac{dx}{dr} + x\frac{dy}{dr},$$

$$\frac{du}{d\theta} = y\frac{dx}{d\theta} + x\frac{dy}{d\theta}.$$

From (2) $$\frac{dx}{dr} = \cos\theta, \quad \frac{dy}{dr} = \sin\theta,$$

$$\frac{dx}{d\theta} = -r\sin\theta, \quad \frac{dy}{d\theta} = r\cos\theta;$$

substituting, $$\frac{du}{dr} = y\cos\theta + x\sin\theta,$$

$$= r,$$

$$\frac{du}{d\theta} = -r^2\sin^2\theta + r^2\cos^2\theta,$$

$$= r^2(\cos^2\theta - \sin^2\theta).$$

Now take the angle $COC' = d\theta$ and $OH = r$, $HC' = dr$, $d\theta$ and dr being small independent increments of θ and r. Complete the parallelogram $OA'C'B'$.

In the partial differential coefficient $\frac{dx}{dr}$, dx is the increment which x would receive if r varied and θ remained constant; or if we make $CQ = dr$, it is the projection of CQ upon OA, which in the limit is the same as $A'K$ which is the projection of HC' upon that line, and is equal to $dr\cos\theta$, and therefore

$$\frac{dx}{dr} = \cos\theta.$$

Again, in the partial differential coefficient $\frac{dx}{d\theta}$, dx is the increment of x when θ varies and r remains constant, and is therefore the projection of HC upon OA or AK taken negatively, since it is measured to the left, and is therefore equal to $- HC \sin\theta = - r \sin\theta\, d\theta$; therefore

$$\frac{dx}{d\theta} = - r \sin\theta.$$

The total differential or actual increment of x due to the increments dr and $d\theta$ of r and θ, is equal to the sum of the partial differentials or to AA'.

In the same manner we may explain the geometrical meanings of dy in the expressions $\frac{dy}{dr}$ and $\frac{dy}{d\theta}$.

In the expression $\frac{du}{dr}$, du represents the increment of the area u due to the increment dr of r; or if the rectangle of which OQ is the diagonal be completed, it will represent the difference between the area of this rectangle and that of AB, which in the limit is equal to

$$y\, dr \cos\theta + x\, dr \sin\theta,$$

therefore
$$\frac{du}{dr} = y \cos\theta + x \sin\theta,$$
$$= r.$$

In the expression $\frac{du}{d\theta}$, du is the increment of the area u due to the increment $d\theta$ of θ, or the difference between the area of the rectangle of which OH is the diagonal and that of AB, which in the limit is equal to

$$xHC \cos\theta - yHC \sin\theta,$$
$$= r(x \cos\theta - y \sin\theta)\, d\theta,$$

therefore
$$\frac{du}{d\theta} = r^2(\cos^2\theta - \sin^2\theta).$$

If we have r and θ given as explicit functions of x and y, the operation for changing the independent variables from x and y to r and θ is as follows.

Let the equations between the variables u, x, y, &c., be

$$u = f(x, y) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$r = \phi(x, y), \quad \theta = \psi(x, y) \ldots\ldots\ldots\ldots (2).$$

From (1) $$\frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx} + \frac{du}{d\theta}\frac{d\theta}{dx},$$

$$\frac{du}{dy} = \frac{du}{dr}\frac{dr}{dy} + \frac{du}{d\theta}\frac{d\theta}{dy};$$

$\frac{dr}{dx}$, $\frac{dr}{dy}$, &c. being obtained from equations (2), $\frac{du}{dx}$ and $\frac{du}{dy}$ may be formed in terms of $\frac{du}{d\theta}$ and $\frac{du}{dr}$.

Applying this method to the same example as in the previous article, we have the equations

$$u = xy,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

$$\frac{du}{dx} = \frac{du}{dr}\frac{dr}{dx} + \frac{du}{d\theta}\frac{d\theta}{dx},$$

$$\frac{du}{dy} = \frac{du}{dr}\frac{dr}{dy} + \frac{du}{d\theta}\frac{d\theta}{dy},$$

$$r\frac{dr}{dx} = x, \quad \therefore \frac{dr}{dx} = \frac{x}{r} = \cos\theta,$$

$$r\frac{dr}{dy} = y, \quad \therefore \frac{dr}{dy} = \frac{y}{r} = \sin\theta,$$

$$(1 + \tan^2\theta)\frac{d\theta}{dx} = -\frac{y}{x^2},$$

$$\therefore \frac{d\theta}{dx} = -\frac{y\cos^2\theta}{x^2} = -\frac{y}{r^2} = -\frac{\sin\theta}{r},$$

$$(1 + \tan^2\theta)\frac{d\theta}{dy} = \frac{1}{x},$$

$$\therefore \frac{d\theta}{dy} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r}.$$

$\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dr}, \frac{du}{d\theta}$ are the same as in the previous article.

In the expression $\frac{dr}{dx}$, dr is the increment of r when x varies and y is constant, or the difference between the lines OD and OC, which is easily seen to be equal in the limit to $dx . \cos\theta$. So in the expression $\frac{dr}{dy}$, dr is the increment of r when y varies and x is constant, or the difference between the lines OE and OC, which is equal to $dy \sin\theta$.

In the expression $\frac{d\theta}{dx}$, $d\theta$ is the increment of θ when x varies and y is constant, or the angle DOC taken negatively, which may be readily shewn to equal $-\frac{dx \sin\theta}{r}$. And similarly we may explain the geometrical meaning of $d\theta$ in the expression $\frac{d\theta}{dy}$.

Implicit Functions.

18. If the variables x and y be connected by an equation

$$f(x, y) = 0,$$

$\frac{dy}{dx}$ is found in the following manner.

Let $$u = f(xy),$$

x and y being supposed for the moment independent; then

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy$$

$$= f'(x) dx + f'(y) dy.$$

Now, introducing the condition that $u = 0$ and therefore that $du = 0$, we have

$$\frac{du}{dx} dx + \frac{du}{dy} dy = 0,$$

or
$$\frac{du}{dx} + \frac{du}{dy}\frac{dy}{dx} = 0;$$

$\frac{du}{dx}$ and $\frac{du}{dy}$ being found on the supposition of x and y being independent.

19. Let us apply this method to the particular example
$$xy - a^2 = 0.$$

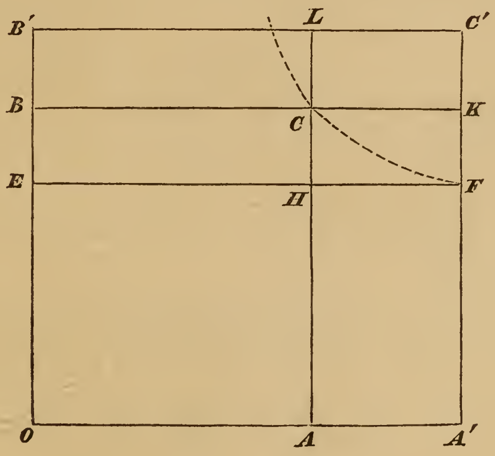
Let
$$u = xy - a^2,$$

therefore
$$du = y\,dx + x\,dy;$$

and, introducing the condition that $u = 0$, and therefore that $du = 0$, we have
$$y\,dx + x\,dy = 0,$$

or
$$y + x\frac{dy}{dx} = 0.$$

To illustrate the geometrical meaning of this process, let OA, OB, the sides of the rectangle AB, be represented by x and y respectively; then u or $xy - a^2$ represents the difference between the area of the rectangle AB and a fixed area a^2.



Let x and y receive the small independent increments AA' and BB' represented by dx and dy respectively. Complete the parallelogram $A'B'$; then in the limit the value of du is the sum of the rectangles $A'C$ and $B'C$. Now let us consider the geometrical meaning of our original equation. It restricts x and y to such values as shall make the area of the rectangle, whose sides are x and y, always equal to a^2, and therefore dx and dy must be consistent with this condition; or dx being supposed given equal to AA', dy must have such a value $-KF$ as shall render the areas of the rectangles AF and BF equal, in order that the area of the new rectangle $A'E$ may be still a^2, or that the increment du shall be zero. This condition gives us in the limit

$$y\,dx = -x\,dy,$$

or

$$y + x\frac{dy}{dx} = 0.$$

We may consider our original equation as the equation to a hyperbola referred to its asymptotes OA and OB as axes, in which case the geometrical meaning of the condition between x and y is that they shall always be such that the angle of the rectangle opposite to O shall always lie in the hyperbola.

20. If u be a function of x and y, so that

$$z = f(x, y) \dots\dots\dots\dots\dots (1),$$

and x and y be connected by an equation

$$y = \phi(x) \dots\dots\dots\dots\dots (2),$$

the differential coefficients of z with respect to x and y are obtained as follows.

In the first place, neglecting equation (2), we have z a function of two *independent* variables x and y; and therefore

$$dz = \frac{dz}{dx}\,dx + \frac{dz}{dy}\,dy,$$

$$= f'(x)\,dx + f'(y)\,dy.$$

Now, introducing the condition involved in equation (2), we have dx and dy connected by the equation

$$dy = \phi'(x)\,dx,$$

and therefore

$$\frac{dz}{dx} = \frac{d_x z}{dx} + \frac{d_y z}{dy}\,\frac{dy}{dx},$$

$\frac{d_x z}{dx}$ and $\frac{d_y z}{dy}$ being found on the supposition of x and y being independent.

Differentiating again,

$$\frac{d^2 z}{dx^2} = \frac{d}{dx}\left(\frac{d_x z}{dx}\right) + \frac{d}{dx}\left(\frac{d_y z}{dx}\,\frac{dy}{dx}\right),$$

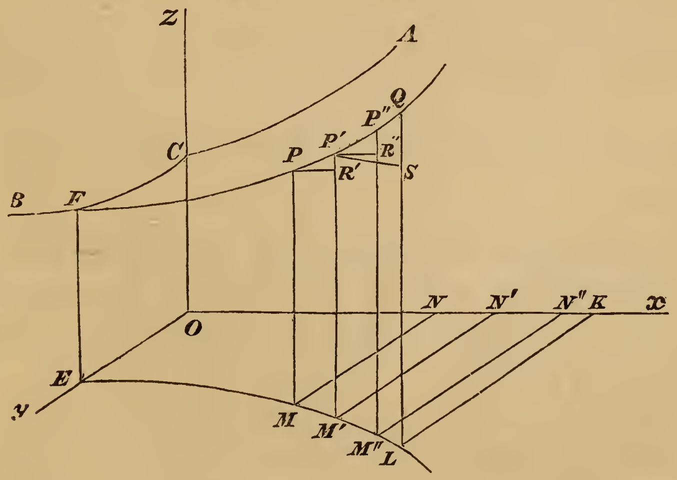
$$= \frac{d_x^{\,2} z}{dx^2} + 2\,\frac{d^2_{\,xy} z}{dx dy}\,\frac{dy}{dx} + \frac{d_y^{\,2} z}{dy^2}\left(\frac{dy}{dx}\right)^2,$$

$\frac{d_x^{\,2} z}{dx^2}$, &c. being found on the same supposition of x and y being independent. In the same manner we may find $\frac{dz}{dy}$ and $\frac{d^2 z}{dy^2}$.

This process may be illustrated as follows.

Let xyz be the coordinates of a point in space referred to a system of rectangular axes; then equation (1), taken by itself, represents a surface, and equation (2) a cylindrical surface whose generating lines are parallel to the axis of z. The two equations taken together represent a curve line, which is the line of intersection of the two surfaces.

Let ACB be the surface represented by equation (1), and



EP the cylindrical surface intersecting the former in the curve line FP and the plane of xy in EM; P any point in this line whose coordinates ON, NM, MP are represented by x, y, z, respectively.

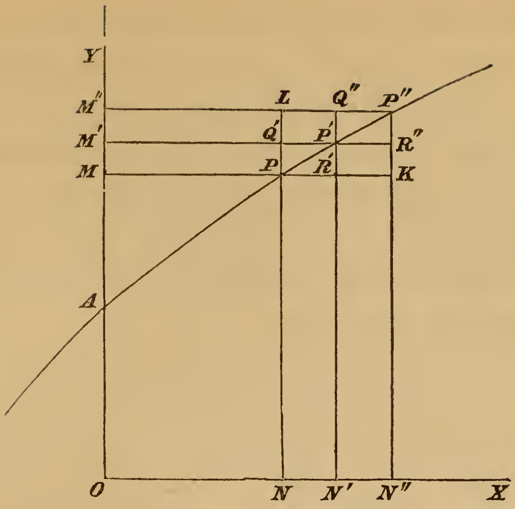
Take $NN' = dx$, and draw $N'M'$ parallel to Oy and cutting the trace of the cylindrical surface on the plane of xy in the point M': draw $M'P'$ cutting the curve line FP in P': draw PR' parallel to the plane of xy and cutting $P'M'$ in R'. $P'M'$ is the new value of z, and therefore the increment $P'R'$ is the total differential dz of z, or the value of dz in the total differential coefficients $\frac{dz}{dx}$ and $\frac{dz}{dy}$.

To explain the meaning of d^2z in the expression $\frac{d^2z}{dx^2}$, take $N'N'' = dx$, and let P'' be the corresponding point in the curve, and draw $P'R''$ parallel to the plane of xy and cutting $P''M''$ in R''. $P''R''$ is the new value of dz, on the supposition of x receiving the increment dx, dx remaining constant. The geometrical representation then of d^2z in $\frac{d^2z}{dx^2}$ is $(P''R'' - P'R')$.

In order to represent d^2z in the expression $\frac{d^2z}{dy^2}$, we must take a point L in the trace FM, such that its ordinate parallel to y, KL shall exceed $M'N'$ by dy: let Q be the corresponding point in the curve, and draw $P'S$ parallel to the plane of xy and cutting QL in S. QS then is the new value of dz on the supposition of y receiving the increment dy, dy remaining constant; and therefore the geometrical representation of d^2z in $\frac{d^2z}{dy^2}$ is $(QS - P'R')$.

The geometrical meanings of dz and d^2z in the partial differential coefficients may be explained in the same manner as for functions of two independent variables, since they are obtained without reference to equation (2).

21. We will now consider the case in which x and y represent the sides ON and NP of the rectangle $ONPM$ and



z its area; when we shall have the equation

$$z = xy \text{ .... } (1),$$

and x and y are connected by an equation

$$y = \phi(x) \text{.................. }(2),$$

which, representing some curve AP, restricts x and y to such values that the angle opposite to O of the rectangle shall lie on the curve.

Differentiating, we have

$$\frac{dz}{dx} = \frac{d_x z}{dx} + \frac{d_y z}{dy}\,\frac{dy}{dx},$$

$$= y + x\,\frac{dy}{dx},$$

$$\frac{d^2z}{dx^2} = 2\,\frac{dy}{dx} + x\,\frac{d^2y}{dx^2},$$

or $$d^2z = 2dy\,dx + x\,d^2y.$$

If we differentiate with respect to y, we shall have

$$\frac{dz}{dy} = x + y\,\frac{dx}{dy},$$

$$\frac{d^2z}{dy^2} = 2\,\frac{dx}{dy} + y\,\frac{d^2x}{dy^2};$$

or $$d^2z = 2dx\,dy + yd^2x.$$

We will now illustrate geometrically the meanings of the two quantities here represented by the same symbol d^2z.

Let $NN' = dx$ be the small increment of x, and $P'Q' = dy$ the corresponding increment of y such that the point P' lies in the curve AP. Complete the rectangle $N'M'$.

The rectangle $N'P$ is the increment which the area z *would* receive, if x varied and y remained constant, which being represented by d_xz, we have

$$d_xz = ydx,$$

or
$$\frac{d_xz}{dx} = y;$$

and similarly $M'P$ is represented by d_yz, so that

$$d_yz = xdy,$$

or
$$\frac{d_yz}{dy} = x.$$

The sum of these is in the limit, the actual increment of z, or the total differential dz, whence

$$dz = ydx + xdy,$$

or
$$\frac{dz}{dx} = y + x\frac{dy}{dx}.$$

Again, let x receive the same increment dx, so that the point P moves to P', and P' to P'', P'' being in the curve, and such that $N'N'' = NN'$. Complete the rectangle $N''M''$.

The sum of the rectangles $P'N''$ and $P'M''$ is the new value of dz, when x receives the increment dx, dx remaining constant; so that the geometrical representation of d^2z in the expression $\frac{d^2z}{dx^2}$ is

$$P'N'' + P'M'' - PN' - PM'$$

$$= P'K + P'L + M'L - PM';$$

and since
$$LQ' - PQ' = d^2y,$$

this equals
$$dy\,dx + (dy + d^2y)\,dx + x\,d^2y;$$

and therefore, neglecting $dx\,d^2y$ as being small compared with the remaining quantities, we have

$$\frac{d^2z}{dx^2} = 2\frac{dy}{dx} + x\frac{d^2y}{dx^2}.$$

To explain the meaning of d^2z in the expression $\frac{d^2z}{dy^2}$, we should proceed in the same manner, except that as this quantity represents the increment of dz, dy remaining constant, we should take the point P'', not such that $N'N''$ shall equal NN', but such that $P''R''$ shall equal $P'R'$ or dy.

The reader will find no difficulty in explaining the meaning of d^2z in the *partial* differential coefficients $\frac{d^2z}{dx^2}$, &c., they being found independently of equation (2), and therefore without reference to the curve AP.

22. The following example will serve to illustrate the manner in which analytical results may be geometrically interpreted.

If $y = f(x)$ be the equation to a curve, then the equation to the normal at the point (xy) is

$$y_1 - y = -\frac{dx}{dy}(x_1 - x),$$

and if (s) be the length of the arc measured from any fixed point,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1,$$

therefore $$2\frac{dx}{ds}\cdot\frac{d^2x}{ds^2} + 2\frac{dy}{ds}\cdot\frac{d^2y}{ds^2} = 0;$$

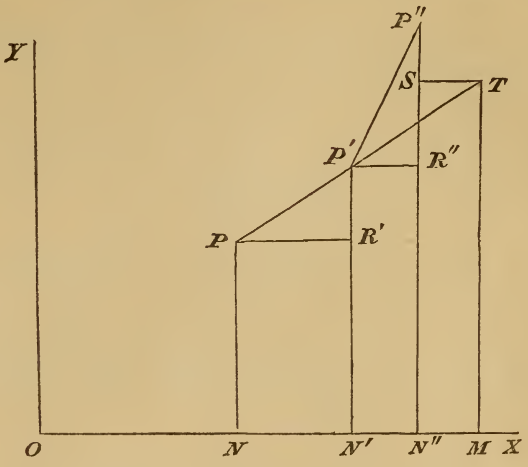
therefore $$-\frac{dx}{dy} = \frac{\dfrac{d^2y}{ds^2}}{\dfrac{d^2x}{ds^2}};$$

and so the equation to the normal may be expressed in the form

$$\frac{x_1 - x}{d^2x} = \frac{y_1 - y}{d^2y} \ldots\ldots\ldots\ldots\ldots (1),$$

d^2x and d^2y being found on the supposition of (s) being the independent variable.

Let P be the point whose coordinates are $ON = x$ and $NP = y$. Take $NN' = dx$, and let P' be the corresponding point in the curve. The straight line PP' may be considered as coinciding with the tangent at P and equal to ds. Let P'' be another point in the curve, such that $P'P'' = PP' = ds$; draw the ordinate $P''N''$. Take $N'M = NN'$, draw the ordinate MT to meet PP' in T; join $P''T$, and draw PR', $P'R''$, and TS parallel to Ox.



Now $P'R' = dy$, and $P''R''$ is what this becomes when s becomes $s + ds$, so that $P''R'' = dy + d_s^2 y$; and since $SR'' = P'R' = dy$, $d_s^2 y = P''S$. Also $N''M = -\,d_s^2 x$, and therefore

$$\tan P''TS = -\frac{d_s^2 y}{d_s^2 x},$$

and the tangent of the angle of inclination of $P''T$ to the axis of x is $\frac{d_s^2 y}{d_s^2 x}$, and the line $P''T$ therefore coincides with the normal at P, since ds is indefinitely small. Equation (1) then is the analytical expression of the fact, that if in the tangent at P, PT be taken equal to the arc PP'', then the line $P''T$ is ultimately perpendicular to the tangent.

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